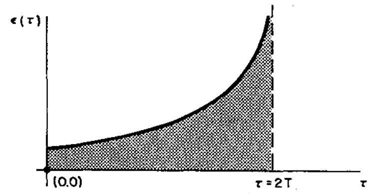
Fig. 3 Graph of error ϵ vs time lag τ



From inequality (13), it is evident that for $\tau_1 < \tau_2$

$$\epsilon(\tau_1) < \epsilon(\tau_2) \tag{14}$$

Now if τ_1 and τ_2 are the maximum values of τ in two separate autocorrelation computations for the same f_1 (t), inequality (14) states that the greater the maximum time lag, the more error is introduced into the final results.

Numerical Example

In conjunction with inequalities (13) and (14), the effect of increasing the maximum time lag τ now will be demonstrated. Thus, let τ_1 be 10% of the total time record on $-T \leqslant t \leqslant T$ and τ_2 be 50% of the same time record. From (13), for $\tau_1 = 0.2T$

$$\tfrac{9}{10}\epsilon(0.2T) \leqslant c/K$$

and for $\tau_2 = T$

$$\tfrac{1}{2}\epsilon(T) \leqslant c/K$$

From these inequalities, it follows that

$$[\tfrac{9}{10}\epsilon(0.2T)]\cdot[-2/\epsilon(T)] \leqslant -1$$

and from (14)

$$\epsilon(0.2T) < \epsilon(T) \leqslant \tfrac{9}{5}\epsilon(0.2T) \tag{15}$$

Since the error introduced by letting the maximum time lag τ be $\frac{1}{2}$ the data record can be almost twice the error introduced by letting the maximum time lag τ be $\frac{1}{10}$ the data record, it is concluded that the maximum lag in digitally computing the autocorrelation function should not exceed 10% of the total data record.

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Orbit-Resonance of Satellites in Librational Motion

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Introduction

A GRAVITATIONALLY oriented satellite executes free rotational oscillations about its mass center at either of two distinct frequencies determined by the mass distribution.[1,2] Since both frequencies are not appreciably greater than the orbital frequency, these are also comparable to the natural frequency of an orbital perturbation. A calculation is presented which shows that the two types of motion, although not dynamically coupled, nevertheless do interact. It is demonstrated that orbital perturbations serve to excite the low-frequency mode of rotary oscillation and that, for orbits of small eccentricity, this occurs at a near-resonant condition.

Stability of Satellite Orbits

Circular orbits are shown in treatises on dynamics to be stable for a class of inverse-power attraction laws which includes Newtonian gravitation. It is shown here first that periodic oscillation about the basic orbit also occurs for noncircular orbits in an inverse-square force field, by considering a small perturbation on an arbitrary "undisturbed" orbit. If $R_0(t)$ and $\vartheta(t)$ denote polar coordinates that locate the mass center of the satellite in its basic undisturbed orbit, and $r'(t)$, $\theta'(t)$ represent the corresponding perturbation quantities, then the "linearized" equations of perturbed motion are

$$\ddot{r}' - R_0\dot{\vartheta}\dot{\theta}' + \dot{\vartheta}^2 r' = (2G/R_0^3)r' \tag{1}$$

and

$$(2\dot{\vartheta}r' + R_0\dot{\theta}')R_0 = \eta \tag{2}$$

where R_0 and ϑ satisfy the equations for the basic orbit, dots denote time differentiation, G is the constant of earth gravitation, and η is an integration constant. Powers and products of disturbance quantities have been neglected systematically in Eqs. (1) and (2), so that these form a system of linear ordinary differential equations. These govern motion of satellite mass center and thus are unaffected by rotary oscillations about that point. Hence they can be analyzed by themselves, and the characteristics of the motion are determined by eliminating the angular variable θ', leaving

$$\ddot{r}' + [2\dot{\vartheta}^2 + (2E/R_0) - (\dot{R}_0^2/R_0^2)]r' = \dot{\vartheta}\eta/R_0 \tag{3}$$

Only for circular orbits are the coefficient and right-hand side both constants, and evaluation of the total energy of the motion E then leads to the equation of the harmonic oscillator at frequency equal to the orbital frequency. Thus, orbital perturbations occur at precisely the rate of 1 cycle/orbit for circular orbits. In the more general case, the coefficients are not constant and the motion is not simple harmonic, but the form of the equation shows that, for orbits of small eccentricity, the "instantaneous" frequencies of orbital perturbations

$$\omega_0 = [2\dot{\vartheta}^2 + (2E/R_0 - \dot{R}_0^2/R_0^2)]^{1/2} \tag{4}$$

are not much different from 1 cycle/orbit. Equations (1) and (2) indicate a 90° phase lag between r' and θ', which will be seen below to have a counterpart in the rotary motions. For simplicity, only circular orbits will be considered henceforth.

Rotational Oscillations about Mass Center (Librations)

Gravity-gradient satellite dynamic characteristics are examined by computing the total moment of momentum of the satellite with respect to its mass center and relating this to the resultant torque moment of gravitational forces acting on constituent mass particles of the body. Orbital oscillations of the type just considered entail angular perturbations θ' that must be included in calculation of moment of momentum. The equation in vector form

$$d\mathbf{h}_0/dt = \mathbf{M}_0 \tag{5}$$

is evaluated for small angular displacements α,β,γ with respect to principal inertia axes, these being shifted only slightly from equilibrium orientation in space. In equilib-

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rium the principal axes x_1, x_2, x_3 are taken, respectively, parallel to orbital angular velocity (and hence perpendicular to orbital plane), oppositely parallel to orbital linear velocity, and radial outward from gravitational center. Orbital angular perturbation thus enhances rotation $\dot{\alpha}$ about x_1, and if the orbital angular velocity is denoted by Ω, the total moment of momentum is evaluated as

$$\mathbf{h}_0 = \mathbf{i}_1(\Omega + \dot{\theta}' + \dot{\alpha})A + \mathbf{i}_2(\dot{\beta} - \Omega\gamma)B + \mathbf{i}_3(\dot{\gamma} + \Omega\beta)C \quad (6)$$

where principal axis unit vectors and moments of inertia A, B, C have been introduced. Note that Ω is the only quantity in parentheses which is not a small perturbation.

In order to evaluate the time derivative of $\mathbf{h}_0$, it is necessary to account for the rotation of coordinate axes and unit vectors resulting from three effects: basic orbital motion Ω, perturbation orbital motion θ', and local rotations having components $\dot{\alpha},\dot{\beta},\dot{\gamma}$. When this is done, one obtains

$$d\mathbf{h}_0/dt = \mathbf{i}_1(\ddot{\alpha} + \ddot{\theta}')A + \mathbf{i}_2\{\ddot{\beta}B + \Omega^2(A - C)\beta + \Omega(A - B - C)\dot{\gamma}\} + \mathbf{i}_3\{\ddot{\gamma}C + \Omega^2(A - B)\gamma - \Omega(A - B - C)\dot{\beta}\} \quad (7)$$

showing that, within the present small-perturbation analysis, orbital perturbation θ' affects only the component of moment of momentum which relates to motion parallel to the orbital plane. (The same is, of course, not true for large-disturbance motion.) When the right-hand side of Eq. (7) is equated to external torque moment, one has an extended form of Euler's rigid body equations, appropriate for motion about a point moving in space.

The gravitational torque moment $\mathbf{M}_0$ vanishes for the equilibrium orientation in space $\alpha = \beta = \gamma = 0$ (indeed, this is the condition that defines equilibrium under the action of gravity gradient forces), and its value has been found for small departures from this orientation to be[2]

$$\mathbf{M}_0 = -3\Omega^2(B - C)\alpha\mathbf{i}_1 - 3\Omega^2(A - C)\beta\mathbf{i}_2 \quad (8)$$

It is evident from the form of (8) that the moment opposes the displacement (static stability) when the two conditions are satisfied:

$$B - C > 0 \qquad A - C > 0 \quad (9)$$

These indicate the only limitations imposed upon the mass distribution and also show which space orientations 90° away from stable equilibrium must be unstable (by interchanging moments of inertia in pairs). The complete dynamic stability is determined by Eqs. (1) and (2), and the three scalar equations obtained by substituting (8) and (9) into (5) are

$$A\ddot{\alpha} + 3\Omega^2(B - C)\alpha = -A\ddot{\theta}' \quad (10)$$

$$B\ddot{\beta} + 4\Omega^2(A - C)\beta + \Omega(A - B - C)\dot{\gamma} = 0 \quad (11)$$

$$C\ddot{\gamma} + \Omega^2(A - B)\gamma - \Omega(A - B - C)\dot{\beta} = 0 \quad (12)$$

for a rigid satellite of arbitrary mass distribution.

Discussion of the Motion

The feature of greatest interest in the present problem is the fact that orbital perturbations affect only the α motions representing oscillations parallel to orbital plane. This occurs only through the term on the right-hand side of Eq. (10), where one may properly regard it as a forcing function for α motion, since the θ' disturbance is found from Eqs. (1) and (2) without regard for Eqs. (10–12). The natural frequency for this principal mode is seen at once to be

$$\omega_\alpha = \Omega 3^{1/2}[(B - C)/A]^{1/2} \quad (13)$$

In the case of greatest practical interest, with axial symmetry such that $A = B$, it is seen that ω_α is never greater than $3^{1/2}$ times the orbital frequency Ω. Since the forcing frequency given by Eq. (4) was seen to be equal to Ω for circular orbits, the natural α motion may be magnified appreciably by the nearness to resonance. Oscillations about x_2 and x_3 axes, given by β and γ, evidently are not affected, since these are coupled with each other but not with the α motion. It is easy to show that, for these symmetric configurations already described, β and γ motions are 90° out of phase with each other, and their frequency then is given by

$$\omega_\beta = \Omega \cdot 2 \cdot [1 - (3C/4A)]^{1/2} \quad (14)$$

Configurations of greatest inherent (static) stability correspond to mass concentrations close to x_3 axis (hence very small values of moment of inertia C); in this limit the frequency given by (14) still is only slightly greater than ω_α (the limiting values, in the ratio $3^{1/2}$:2, have been given correctly by Domojilova and her co-workers in the Russian literature[1]).

Conclusions

The characteristic rotational motions of gravity-gradient satellites do not affect the stability or the period of oscillations due to orbital disturbances, nor does the rotational motion induce an orbital perturbation. The converse is not true: orbital oscillations affect the rotational motion parallel to orbital plane, and the interaction is in the nature of an external forcing function that is sinusoidal. The closeness of forcing frequency to system natural frequency ω_α will require closer examination, particularly for eccentric orbits.

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Bow Shock Correlation for Slightly Blunted Cones

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When solving for the flow properties in the re-entry trail of a hypersonic nose cone, it is in general necessary first to solve for the detailed flow field in the vicinity of the body. However, in some cases (e.g., for relatively short bodies at high altitudes), and for the purposes of a parametric analysis, it is sufficient merely to specify the shape of the bow shock.[1,6] Thus, if a simple but accurate correlation can be used, based only on body geometry and freestream conditions, a considera-able saving in expense and effort will result.

For highly blunted bodies (e.g., a hemisphere cylinder), the Van Hise correlation[2] based on the blast wave analogy yields good results. The equation for the shock shape in this case is

$$r/r_n = 1.424[C_D{}^{0.5}(x/r_n)]^{0.46} \quad (1)$$

where r and x are cylindrical polar coordinates, with x measured along the body axis, whereas r_n is the base radius of the body, and C_D is its drag coefficient. It has been found, however, that, for slightly blunted conical nose shapes of low drag coefficient, the Van Hise correlation is not satisfactory. For these cases, a modified correlation has been formulated.

Small angle conical nose shapes, capped with a spherical segment of radius r_T, with $r_T/r_n \ll 1$, will be considered.

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